SOME ALGEBRAS SIMILAR TO THE 2×2 JORDANIAN MATRIX ALGEBRA

JASON GADDIS AND KENNETH L. PRICE

Abstract. The impetus for this study is the work of Dumas and Rigal on the Jordanian deformation of the ring of coordinate functions on 2×2 matrices. We are also motivated by current interest in birational equivalence of noncommutative rings. Recognizing the construction of the Jordanian matrix algebra as a skew polynomial ring, we construct a family of algebras relative to differential operator rings over a polynomial ring in one variable which are birationally equivalent to the Weyl algebra over a polynomial ring in two variables.

1. Introduction

In the study of quantum groups, one wishes to understand algebras (or their representation theory) which arise through various constructions, be it from the Yang-Baxter equation, Hopf actions, or quantum enveloping algebras. Our goal is contribute to the understanding of algebras that lie slightly outside of the study of quantum groups. That is, to study those algebras which are similar to quantum groups in some sense. The classic example here is the Weyl algebra, or any of its quantum analogs. One could approach this problem by looking at PBW deformations of quantum groups, or by introducing parameters into one of the constructions listed above.

In this work, we will be primarily interested in the 2×2 quantum matrix algebras, with particular interest in the matrix algebra corresponding to the Jordan plane, denoted $M_J(2)$. We review relevant definitions in Section 2. Our goal is to put $M_J(2)$ inside a larger class of algebras which we call *Generalized Jordanian Matrix Algebras* (GJMAs). These are algebras birationally equivalent to $M_J(2)$ in the sense that they have the same quotient ring of fractions. They also maintain a specific involution which can be thought of as the transposition operator on 2×2 matrices. We construct a family of GJMAs as iterated skew polynomial rings. Much like the universal enveloping algebra of sl_2 and its generalizations [18], GJMAs are distinguished by a degree two central element, which can also be thought of as a generalization of the determinant in $M_J(2)$.

In Section 3, we review the definition of skew polynomial rings, with an emphasis on those of differential operator type. We introduce in Section 4 a class of algebras which we call *involutive skew polynomial extensions* (ISPE), which generalizes the base rings of GJMAs. These are two-step skew polynomial extensions with an automorphism fixing the base ring and interchanging the two new generators. It is shown that under certain conditions an ISPE over a ring R is birationally equivalent to the first Weyl algebra over R.

2010 *Mathematics Subject Classification.* 16S36, 16S85.

The notion of an ISPE is specialized in Section 5 to those with base a polynomial ring in one variable. Such algebras form the base ring of $M_J(2)$ as well as the quantum matrix algebras $M_q(2)$. We classify these ISPEs in Propositions 5.1, 5.3, and 5.5.

Section 6 specializes further to differential operator rings of the form $\Bbbk[c][a;\delta_1]$. This class includes the base ring of the Jordanian matrix algebra, and in general certain elements of this class serve as the base ring of GJMAs. These algebras are interesting in their own right from a representation theory point of view. In particular, they merge the limited class of 1-dimensional representations coming from differential operator rings over $\Bbbk[c]$ with the large class of n-dimensional representations corresponding to a polynomial ring in two variables.

Finally, in Section 7, we introduce and study GJMAs. If P is an ISPE over $\Bbbk[c]$ of differential operator type discussed above and $M = P[b;\sigma_3,\delta_3]$ such that, under some localization of M, δ_3 is an inner σ_3-derivation and σ_3 an inner automorphism, then M is a birationally equivalent to $M_J(2)$ (Proposition 7.4). We construct a specific class of such algebras and prove that they are GJMAs (Proposition 7.6) containing a central element which can be thought of as an analog of the determinant. Like $M_J(2)$, these algebras are noetherian domains of GK and global dimension four (Proposition 7.9). The remainder of the work is devoted to a study of the prime ideals of such GJMAs.

2. Quantum Matrix Algebras

Throughout, $\Bbbk$ is an algebraically closed, characteristic zero field and all algebras are $\Bbbk$-algebras. Isomorphisms should be read as 'isomorphisms as $\Bbbk$-algebras'. All unadorned tensor products should be regarded as over $\Bbbk$.

For $q \in \Bbbk^\times$, the quantum plane is the $\Bbbk$-algebra $\mathcal{O}_q(\Bbbk^2) = \Bbbk\langle x_1, x_2 \mid x_1x_2 - qx_2x_1\rangle$. Classically, the 2×2 *quantum matrix algebra* relative to q, sometimes denoted $\mathcal{O}_q(M_2(\Bbbk))$, is the unique $\Bbbk$-algebra on generators x_{11}, x_{12}, x_{21}, x_{22} such that there exist homomorphisms

$$\begin{aligned} \mathcal{O}_q(\Bbbk^2) &\to M_q(2)\otimes\mathcal{O}_q(\Bbbk^2) & \quad\text{and}\quad & & \mathcal{O}_q(\Bbbk^2) &\to \mathcal{O}_q(\Bbbk^2)\otimes M_q(2) \\ x_i &\mapsto x_{i1}\otimes x_1 + x_{i2}\otimes x_2 & & & x_j &\mapsto x_1\otimes x_{1j} + x_2\otimes x_{2j}. \end{aligned}$$

Making the identifications $b = x_{11}$, $c = x_{22}$, $a = x_{12}$, $d = x_{21}$, we get the relations

$$\begin{aligned} ac &= qca & \qquad dc &= qcd & \qquad da &= ad \\ ba &= qab & bd &= qdb & bc &= cb + (q-q^{-1})ad. \end{aligned}$$

In this way, $M_q(2)$ may be regarded as a deformation of the coordinate ring of functions on 2×2 matrices. For more details on this construction, the reader is encouraged to see [5], [12], and [16].

There are two candidates for deforming the Jordan plane, $\mathcal{J} = k\langle x_1, x_2 \mid x_2x_1 - x_1x_2 + x_2^2\rangle$. The one presented in [16] is not a domain. We prefer the one presented in [6] which may be constructed as above by

replacing $\mathcal{O}_q(\Bbbk^2)$ with $\mathcal{J}$. This Jordanian matrix algebra, denoted $M_J(2)$ or $\mathcal{O}_J(M_2(\Bbbk))$, is the $\Bbbk$-algebra on generators a, b, c, d subject to the following relations.

$$
\begin{aligned}
ac &= ca + c^2 & \qquad bc &= cb + ca + cd + c^2 \\
dc &= cd + c^2 & \qquad bd &= db + cb + cd - ad + d^2 \\
da &= ad - cd + ca & \qquad ba &= ab + cb + cd - ad + a^2.
\end{aligned}
$$

The algebra $M_J(2)$ is a domain and can be constructed as a skew polynomial ring. Dumas and Rigal studied the prime spectrum and automorphisms of this algebra in [9], though the presentation they give is slightly different (but equivalent) to the one given here.

Remark 2.1. *Letting $u = d - a$, we have the alternate presentation of $M_J(2)$.*

$$
\begin{aligned}
ac &= ca + c^2 & \qquad bc &= cb + c(2a + u + c) \\
uc &= cu & \qquad bu &= ub + u(2a + u + c) \\
ua &= au - cu & \qquad ba &= (a + c)b + (c - u)a.
\end{aligned}
$$

Remark 2.2. *The formulas $\tau(a) = d$, $\tau(d) = a$, $\tau(c) = c$, and $\tau(b) = b$ determine either an automorphism $\tau : M_J(2) \to M_J(2)$ or $\tau : M_q(2) \to M_q(2)$. One should think of this operation as the transposition operator on 2×2 matrices. For either $M_q(2)$ or $M_J(2)$ the automorphism group G is the semidirect product $H \rtimes \{\tau\}$ for some subgroup H of G. In the case of $M_q(2)$, we have $H \cong (\Bbbk^\times)^3$* [1, Theorem 2.3]. *In the case of $M_J(2)$, H is considerably more complex and the interested reader is referred to* [9, Proposition 3.1].

In Section 7 we construct a family of $\Bbbk$-algebras on generators a, b, c, u with relations

$$
\begin{gathered}
ac = ca + cg, cu = uc, au = ua + ug \\
bc = cb + c\gamma, bu = ub + u\gamma, ba = (a + h)b + (h - u)a,
\end{gathered}
$$

where $g \in \Bbbk[c]$, $h = cg'$, and $\gamma = h + u + 2a$. We denote such an algebra by $\mathcal{M}_f$, where $f = cg$. Note that $\mathcal{M}_{c^2}$ gives the alternate presentation for $M_J(2)$ in Remark 2.1.

The centers of $M_q(2)$ and $M_J(2)$ are each generated by a single degree two element, known as the *quantum determinant*. In the case of $M_q(2)$, the quantum determinant is $bc - qad$, and for $M_J(2)$ the quantum determinant is $ad - cb - cd$. The element $z = gb + (g - u)a - a^2$ is central in $\mathcal{M}_f$.

If $M = M_q(2)$ or if $M = M_J(2)$ and z is the quantum determinant of M, then $M/(z-1)$ is a deformation of the coordinate ring of functions on SL_2. In particular, $M/(z - 1)$ is a *noncommutative quadric* [19]. We can similarly construct such factor rings when M is a GJMA as constructed above. In this way, we view $\mathcal{M}_f/(z - 1)$ as a sort of nonhomogeneous noncommutative quadric.

3. Skew polynomial rings and Weyl algebras

Let R be a ring. The first Weyl algebra over R, denoted $A_1(R)$, is the overring of R with additional generators x and y which commute with R and satisfy $xy = yx + 1$.

Let σ be an automorphism of R and let δ be a σ-derivation, that is, $\delta : R \to R$ is a $\Bbbk$-linear map such that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all $r, s \in R$. The *skew polynomial ring* (or Ore extension) $R[x; \sigma, \delta]$ is defined via the commutation rule $xr = \sigma(r)x + \delta(r)$ for all $r \in R$. If $\delta = 0$, then we omit it and write $R[x; \sigma]$. If σ is the identity, then we omit it and write $R[x; \delta]$. A skew polynomial ring of this last form is called a *differential operator ring* over R.

A σ-derivation δ of R is said to be σ-*inner* if there exists $\theta \in R$ such that $\delta(r) = \theta r - \sigma(r)\theta$ for all $r \in R$. In this case, the skew polynomial ring $R[x; \sigma, \delta]$ is equivalent to $R[x - \theta; \sigma]$. An automorphism σ of R is said to be *inner* if there exists a unit $\varphi \in R$ such that $\varphi^{-1} r \varphi = \sigma(r)$ for all $r \in R$. In this case, the skew polynomial ring $R[x; \sigma, \delta]$ is equivalent to $R[\varphi x; \varphi\delta]$.

Let σ_1 be an automorphism of $\Bbbk[c]$ and δ_1 a σ_1-derivation. Write $A = \Bbbk[c][a; \sigma_1, \delta_1]$. By [4, Remark 2.1], A is one of the following:

- Quantum plane: $\mathcal{O}_q(\Bbbk^2) = \Bbbk\langle a, c \mid ac - qca\rangle$, $q \in \Bbbk^\times$;
- Quantum Weyl algebra: $A_1^q(\Bbbk) = \Bbbk\langle a, c \mid ac - qca - 1\rangle$, $q \in \Bbbk^\times$;
- Differential operator ring: $R_f = \Bbbk\langle a, c \mid ca - ac + f\rangle$, $f \in \Bbbk[c]$.

There is no overlap between the different classes except that the first Weyl algebra $A_1(\Bbbk)$ over $\Bbbk$ is both a quantum Weyl algebra with $q = 1$, and a differential operator ring with $f = 1$. See [10] and [11] for further study on isomorphisms within each class.

Our work will focus heavily on the class of differential operator rings above. These algebras have been studied extensively by various authors. For a good overview, see [12]. We review some basic facts about these algebras here.

By [11, Theorem 1.1], R_f is isomorphic to either $A_1(\Bbbk)$, the universal enveloping algebra of the 2-dimensional solvable Lie algebra (R_c), the Jordan plane (R_{c^2}), or the deformed Jordan plane (R_{c^2-1}). For all $f \in \Bbbk[c]$, the differential operator ring R_f is a noetherian domain of global and GK dimension two. In characteristic zero, the center of R_f is $\mathcal{Z}(R_f) = \Bbbk$.

Denote by $\mathrm{Frac}(R)$ the clasical quotient ring of R. Two rings R and S are said to be *birationally equivalent* if $\mathrm{Frac}(R) \cong \mathrm{Frac}(S)$. We will make use of additional properties for differential operator rings from [9, Proposition 1.8] and [3, Proposition 2.6]. We state these results below using our notation.

Proposition 3.1 (Alev and Dumas)**.** *Choose $f, g \in \Bbbk[c]$, $f, g \neq 0$.*

(1) *The differential operator ring R_f is birationally equivalent to $R_1 = A_1(\Bbbk)$.*
(2) *$R_f \cong R_g$ if and only if there exists $\lambda, \alpha \in \Bbbk^\times$ and $\beta \in \Bbbk$ such that $f(c) = \lambda g(\alpha c + \beta)$.*

(3) *For any $\alpha, \lambda \in \Bbbk^\times$ and $\beta \in \Bbbk$ there is an automorphism of R_f such that $f(\alpha c + \beta) = \alpha\lambda f(c)$ determined by the assignments $a \mapsto \lambda a - g$ and $c \mapsto \alpha c + \beta$. Moreover, when $f \notin \Bbbk$ (i.e., $R_f \ncong A_1(\Bbbk)$) any automorphism of R_f has the above form.*

We conclude this section with automorphisms of the first Weyl algebra, $A_1(\Bbbk) = \Bbbk\langle a, c \mid ac = ca + 1\rangle$. Diximier [8] gave generators for $\mathrm{Aut}(A_1(\Bbbk))$, but we only need the limited subgroup of automorphisms detailed in the following proposition.

Proposition 3.2. *If $\sigma \in \mathrm{Aut}(A_1(\Bbbk))$ satisfies $\sigma(c) = c$, then $\sigma(a) = a - g$ for some $g \in \Bbbk[c]$.*

Proof. Write $\sigma(a) = \sum_{i=0}^n a^i g_i$ for $g_i \in \Bbbk[c]$. Then

$$1 = \sigma([a, c]) = [\sigma(a), c] = \left[\sum_{i=0}^n a^i g_i, c\right] = \sum_{i=0}^n [a^i, c] g_i = \sum_{i=0}^n i a^{i-1} g_i.$$

Consequently, $g_i = 0$ for $i > 1$, $g_1 = 1$, and g_0 is arbitrary. Set $g = -g_0$. □

4. Involutive skew polynomial extensions

Definition 4.1. *Let R be a ring. We say $S = R[a; \sigma_1, \delta_1][d; \sigma_2; \delta_2]$ is an **involutive skew polynomial extension** (ISPE) of R if there exists an involution $\tau \in \mathrm{Aut}(S)$ such that $\tau(a) = d$, $\tau(d) = a$, and $\tau(r) = r$ for all $r \in R$.*

Remark 4.2. *One possible generalization of this definition is to requires S to be a* **double extension** *of R as defined by Zhang and Zhang* [20, 21]. *Because our primary interest is in (ordinary) skew polynomial rings, we do not make that definition here.*

It is well known that such an S is a noetherian domain if R is. We give several examples of ISPEs below.

Example 4.3. *When $R = \Bbbk$, $R[a; \sigma_1, \delta_1]$ is a polynomial extension and so $\sigma_1 = \mathrm{id}_R$ and $\delta_1 = 0$. Then S is generated by a and d subject to the single relation $0 = qad - da + f(a)$ for some $q \in \Bbbk^\times$ and $f(a) \in \Bbbk[a]$. Applying the involution τ gives*

$$0 = qda - ad + f(d) = q(qad + f(a)) - ad + f(d) = (q^2 - 1)ad + (qf(a) + f(d)).$$

Hence $q^2 = 1$ and f is a constant polynomial. This implies that either $S = \mathcal{O}_q(\Bbbk^2)$ with $q = \pm 1$ or $S = A_1^q(\Bbbk)$ with $q = -1$.

Let S be an ISPE of a ring R. The element $u = d - a$ of S appears in [9] and is critical to the study of birational equivalence in ISPEs. Using the involution τ, one can show $\sigma_1(r) = \sigma_2(r)$, $\delta_1(r) = \delta_2(r)$, and hence $ur = \sigma_1(r)u$ for all $r \in R$.

Proposition 4.4. *Let S be an ISPE of a ring R. The element $u = d - a$ is normal in S if and only if $\delta_2(a) = a^2 - \sigma_2(a)a$.*

Proof. We have already seen that $ur = \sigma_1(r)u$ for all $r \in R$. Moreover, the d-degree of $\delta_2(a) - a^2$ is zero in

$$(d-a)a = \sigma_2(a)d + \delta_2(a) - a^2.$$

Thus, $(d-a)$ is normal if and only if $(d-a)a = \sigma_2(a)(d-a)$ if and only if $\delta_2(a) - a^2 = -\sigma_2(a)a$. $\square$

In this case, we have an alternate presentation of S with relations

$$\begin{aligned} &ar = \sigma_1(r)a + \delta_1(r) \text{ and } ur = \sigma_1(r)u \text{ for all } r \in R, \\ &ua = \sigma_2(a)u. \end{aligned}$$

Proposition 4.5. *Let S be an ISPE of R and suppose the following conditions hold:*

- *R is affine and commutative with generators $r_1, \ldots, r_n$ and $\sigma_1 = \mathrm{id}_R$.*
- *u is normal in S.*
- *$r_i^{-1}\delta_1(r_i) = r_j^{-1}\delta_1(r_j)$ for all $i, j \in \{1, \ldots, n\}$.*
- *$a\left(r_i^{-1}u\right) = \left(r_i^{-1}u\right)a$.*

Then S is birationally equivalent to $A_1(R)$.

Proof. Let $Q = \mathrm{Frac}(S)$ and let x, y, t_i be the standard generators of $A_1(R)$. Let $f = r_1^{-1}\delta_1(r_1)$. Because u commutes with R, the elements $r_i^{-1}u$ in Q generate an isomorphic copy of R. We associate these elements with the standard generators t_i of R in $A_1(R)$. Hence, by abuse of notation, f may simultaneously be thought of as an element of R in the generators r_i and in the t_i. It now follows by the hypotheses that there are inverse homomorphisms $\Phi : Q \to \mathrm{Frac}(A_1(R))$ and $\Psi : \mathrm{Frac}(A_1(R)) \to Q$ given by

$$\begin{aligned} &\Phi(a) = xf, \Phi(u) = yt_1, \Phi(r_i) = t_i^{-1}yt_1, \\ &\Psi(x) = af^{-1}, \Psi(y) = r_1, \Psi(t_i) = r_i^{-1}u. \end{aligned}$$

$\square$

Example 4.6. *Let R be any affine commutative ring with generators $r_1, \ldots, r_n$. Let S be the extension of R with relations $ar_i = r_ia + r_i$, $dr_i = r_id + r_i$, and $da = (a-1)d + a$. Then S is an ISPE of R. By Proposition 4.5, S is birationally equivalent to $A_1(R)$.*

Example 4.7. *Let $R = \Bbbk[c]$, $\sigma_1(c) = \sigma_2(c) = c$, $\delta_1(c) = \delta_2(c) = c^2$, $\sigma_2(a) = (a-c)$, and $\delta_2(a) = ca$. Then S is an ISPE. In particular, S is the base ring in the skew polynomial construction of $M_J(2)$. By Proposition 6.12, S is birationally equivalent to $A_1(\Bbbk[t])$.*

5. Classification of ISPEs over $\Bbbk[c]$

Throughout this section, let $A = \Bbbk[c][a; \sigma_1; \delta_1]$ and $P = A[d; \sigma_2; \delta_2]$.

Proposition 5.1. *If P is an ISPE with $A = A_1^q(\Bbbk)$ and $q \neq 1$, then $q = -1$ and P has relations $ac + ca = 1$, $dc + cd = 1$, $da + ad = h$ with $h \in \Bbbk[c]$.*

Proof. We have $A = \Bbbk\langle a, c \mid ac - qca - 1\rangle$. Let $\sigma = \sigma_2$ and $\delta = \delta_2$. The existence of the involution τ implies $dc = qcd + 1$. In particular, $\sigma(c) = qc$ and $\delta(c) = 1$. Because σ is an automorphism of $A_1^q(\Bbbk)$, then $\sigma(a) = q^{-1}a$ [2, Proposition 1.5] giving the final relation

$$da = q^{-1}ad + \delta(a). \tag{1}$$

Applying τ to (1) gives $ad = q^{-1}da + \tau(\delta(a))$, or equivalently,

$$da = qad - q\tau(\delta(a)). \tag{2}$$

Combining (1) and (2) yields $(q - q^{-1})ad = \delta(a) + q\tau(\delta(a))$. Since the d-degree (resp. a-degree) of $\delta(a)$ (resp. $\tau(\delta(a))$) is zero, then $(q - q^{-1})ad = 0$, implying $q = \pm 1$.

If $q = -1$, then we are left only to determine $\delta(a)$. The above computation shows that $\delta(a)$ is τ-invariant. Thus, $\delta(a) \in \Bbbk[c]$. We need only show that any choice of polynomial in $\Bbbk[c]$ produces a valid σ-derivation.

$$\begin{aligned}\delta(ac + ca - 1) &= (\sigma(a)\delta(c) + \delta(a)c) + (\sigma(c)\delta(a) + \delta(c)a) \\ &= (-a + \delta(a)c) + (-c\delta(a) + a) = \delta(a)c - c\delta(a).\end{aligned}$$

Thus, δ is a σ-derivation if $\delta(a)$ commutes with c. □

Remark 5.2. *Set $h = -2$ in Proposition 5.1 and let I denote the ideal generated by c^2, a^2, and d^2, which are all central elements of P in this case. The elements $x_1 = a+c+I$, $x_2 = c+d+I$, and $x_3 = \sqrt{-1/2}(a+d)+I$ of the factor ring P/I satisfy the following:*

$$x_i x_j = \begin{cases} -x_j x_i & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

It is easy to see that P/I is isomorphic to the Clifford algebra of a 3-dimensional regular quadratic space over $\Bbbk$. (A good reference for Clifford algebras is [14]*.)*

We now proceed to the quantum plane case.

Proposition 5.3. *Choose $q \in \Bbbk^\times$ such that $q \neq 1$ and let $A = \mathcal{O}_q(\Bbbk^2)$. There are two possible ISPEs $P = A[d; \sigma_2, \delta_2]$ with common relations $ac = qca$, $dc = qcd$, and either $da = ad$ or $da + ad = h$ for some $h \in \Bbbk[c]$.*

Proof. We have $A = \Bbbk\langle a, c \mid ac - qca\rangle$. Let $\sigma = \sigma_2$ and $\delta = \delta_2$. The existence of the involution τ implies $dc = qcd$ so that $\sigma(c) = qc$ and $\delta(c) = 0$. Moreover, since σ is an automorphism of A, $\sigma(a) = pa$ for some $p \in \Bbbk^\times$ [1, Proposition 1.4.4]. We have

$$\delta(a) = \sum_{j=0}^{m} h_j a^j$$

for some $h_0, \dots, h_m \in k[c]$. Applying τ to the commutation relation $da = pad + \delta(a)$ gives

$$ad = pda + \sum_{j=0}^{m} h_j d^j$$

since $\tau(h_j) = h_j \in k[c]$ for all j. Combining this with $da = pad + \delta(a)$ gives

$$ad = p^2 ad + p \sum_{j=0}^{m} h_j a^j + \sum_{j=0}^{m} h_j d^j$$

and equating coefficients of d on both sides of the equation gives $p^2 = 1$.

If $p = 1$, then $h_j = 0$ for all j, and $\delta(a) = 0$. On the other hand, if $p = -1$, then $h_j = 0$ for all $j \geq 1$. $\square$

Remark 5.4. *The above arguments yield the same result if we replace τ with a sign-changing automorphism σ such that $\sigma(a) = -d$ and $\sigma(d) = -a$. This is reminiscent of the distinction between Lie algebra and Lie superalgebra.*

Recall the algebra $R_f = \mathbb{k}\langle a, c \mid ac = ca + f\rangle$ is defined for any $f \in \mathbb{k}[c]$. To construct ISPEs in this case, we describe R_f as an Ore extension, $R_f = \mathbb{k}[c][a; \delta_1]$ with $\delta_1(c) = f$. An easy induction argument shows $ac^n = c^n a + nfc^{n-1}$ for all $n \geq 0$. This implies $ah = ha + fh'$ and $\delta_1(h) = f\partial_c(h) = fh'$, where $\partial_c(h) = h'$ is the usual derivative of h.

Proposition 5.5. *Suppose $f \in k[c]$ and $R_f[d; \sigma_2; \delta_2]$ is an ISPE. There exists $g \in k[c]$ such that $\sigma_2(c) = c$, $\delta_2(c) = f$, $\sigma_2(a) = a - g$, and $\delta_2(a) = ga$. Thus we may denote $R_f[d; \sigma_2, \delta_2]$ by $P(f, g)$ since it depends only on f and g.*

Proof. Applying τ to $ac = ca + f$ gives $dc = cd + f$ so $\sigma_2(c) = c$ and $\delta_2(c) = f$. We find $\sigma_2(a) = a - g$ by Proposition 3.1 ($\deg f > 1$), Proposition 3.2 ($f \in \mathbb{k}$) and [7, Theorem 2] if $f = 0$. We check the formula for $\delta_2(a)$.

Recall $\delta_1(h) = fh'$ for all $h \in \mathbb{k}[c]$. A similar argument applies to δ_2 since $\delta_2(c) = \delta_1(c) = f$. This shows $\delta_2(h) = \delta_1(h) = fh'$ for all $h \in \mathbb{k}[c]$. Applying δ_2 to both sides of $ac = ca + f$ and reducing leads to the formula $\delta_2(a)c = c\delta_2(a) + gf$.

We have $\delta_2(a) \in R_f$ so $\delta_2(a) = \sum_{j=0}^{m} h_j a^j$ for some $h_0, \dots, h_m \in \mathbb{k}[c]$. We substitute this formula into $\delta_2(a)c = c\delta_2(a) + gf$.

$$\begin{aligned} gf = \delta_2(a)c - c\delta_2(a) &= \sum_{j=0}^{m} h_j \left(a^j c - ca^j\right) = \sum_{j=0}^{m} h_j \left(\left(ca^j + \sum_{i=1}^{j} \binom{j}{i} \left(\delta_1(c)\right)^i a^{j-i} \right) - ca^j \right) \\ &= \sum_{j=1}^{m} \sum_{i=1}^{j} h_j \binom{j}{i} \left(\delta_1(c)\right)^i a^{j-i} = \sum_{x=0}^{m-1} \left(\sum_{y=1}^{m-x} \binom{x+y}{y} h_{x+y} (\delta_1(c))^y \right) a^x. \end{aligned}$$

We have $gf \in \mathbb{k}[c]$ so $gf = \sum_{y=1}^{m} h_y \left(\delta_1(c)\right)^y$ and $\sum_{y=1}^{m-x} \binom{x+y}{y} h_{x+y} \left(\delta_1(c)\right)^y = 0$ for all x, $1 \leq x \leq m-1$. Setting $x = m-1$ gives $mh_m \delta_1(c) a^{m-1} = 0$ so $h_m = 0$. Setting $x = m-2$ gives $(m-1)h_{m-1}\delta_1(c) = 0$ so

$h_{m-1} = 0$. Continuing in this fashion, we find $h_\ell = 0$ for all $\ell = 2, \dots, m$. This gives $gf = h_1\delta_1(c) = h_1 f$ so $g = h_1$. Moreover, $\delta_2(a) = h_0 + h_1 a$ so $\delta_2(a) = h_0 + ga$.

Our arguments have lead us to the relation $da = (a-g)d + h_0 + ga$. Applying τ and reducing gives $h_0 = 0$. We are left with $da = (a-g)d + ga$ and $\delta_2(a) = ga$, as desired. □

Example 5.6. *The base ring of the Jordanian matrix algebra $M_J(2)$ is the ISPE $P(c^2, c)$.*

6. The algebras $P(f, g)$

In this section we consider the properties of the ISPEs over $\Bbbk[c]$ relative to the differential operator rings R_f, including presentations, prime spectrum, and birational equivalence.

Remark 6.1. *An easy check shows $\delta_2(c) = ac - \sigma_2(c)a$ and $\delta_2(a) = a(a) - \sigma_2(a)a$ in $P(f, g)$. Thus, δ_2 is the inner σ_2-derivation induced by a.*

Remark 6.2. *The presentation of $M_J(2)$ determined by setting $u = d-a$ extends to alternative presentations of $P(f, g)$ (see Remark 2.1). We have $du = u(d+g)$, $ac = ca + f$, $uc = cu$, and $au = ua + gu$.*

Presentation 1: $P(f, g) = \Bbbk[c][a; \delta_1][u; \sigma_2]$.

Presentation 2: $P(f, g) = \Bbbk[c, u][a; D]$ *where D is the derivation $f\partial_c + gu\partial_u$.*

A (two-sided) ideal I of R_f is prime if and only if $I \cap \Bbbk[c]$ is δ-invariant. That is, if $J = I \cap \Bbbk[c]$, then $\delta_1(J) \subseteq J$. Thus, if I is a prime ideal of R_f, then I is generated by an irreducible factor of f. Hence, since $\Bbbk$ is algebraically closed, $I = (c - \alpha)$ for some $\alpha \in \Bbbk$ such that $f(\alpha) = 0$.

Proposition 6.3. *Choose $f, g \in \Bbbk[c]$ and set $P = P(f, g)$. The following elements of P are normal: f, $u = d - a$, and $c - \alpha$ for any $\alpha \in \Bbbk$ such that $f(\alpha) = 0$. Moreover, the ideals generated by (i) u, (ii) $c - \alpha$, and (iii) both u and $c - \alpha$ are prime in P.*

Proof. Denote the ideals described in (i), (ii), and (iii) of (1) by I_1, I_2, and I_3, respectively.

We showed u is normal in Remark 6.2. Suppose $f \notin \Bbbk$ and note that f is a product of its linear factors since $\Bbbk$ is algebraically closed. If $\alpha \in \Bbbk$ such that $f(\alpha) = 0$, then there exists $f_0 \in \Bbbk[c]$ such that $f = (c-\alpha)f_0$ and $c(c-\alpha) = (c-\alpha)c$, $a(c-\alpha) = (c-\alpha)(a + f_0)$, and $d(c-\alpha) = (c-\alpha)(d + f_0)$ so $c - \alpha$ is normal.

Then $P/I_1 \cong R_f$ and $P/I_3 \cong \Bbbk[a]$, so I_1 and I_3 are prime. Moreover,

$$P/I_2 \cong \begin{cases} k[a, d] & \text{if } g(\alpha) = 0 \\ U(L) & \text{if } g(\alpha) \neq 0, \end{cases}$$

where $U(L)$ is the universal enveloping algebra of the two-dimensional solvable Lie algebra L. Hence I_2 is also prime. □

Definition 6.4. *Let $f, g \in \Bbbk[c]$, $p_1, \dots, p_k$ the irreducible factors of f and $f_i = p_i^{-1} f$. We say g* ***is a local reduction of*** *f if there exists integers $n, m_1, \dots, m_k$ ($n \neq 0$) such that*

$$g = \begin{cases} 0 & \text{if } f \in \Bbbk \\ -\frac{1}{n} \sum_{i=1}^{k} m_i f_i & \text{otherwise.} \end{cases}$$

Proposition 6.5. *Let $P = P(f, g)$ and let $\mathcal{C}$ be the Ore set generated by powers of the irreducible factors $p_1, \dots, p_k$ of f. Let $\mathcal{R} = R_f \mathcal{C}^{-1}$. Then $Q = \mathcal{R}[u^{\pm 1}; \sigma_2]$ is simple if and only if $f \in \Bbbk$ or g is not a local reduction of f.*

Proof. It follows from Proposition 3.1 that $\mathcal{R}$ is a simple ring. Thus by [17, Theorem 1.8.5], Q is simple if and only if no power of σ_2 is inner. Set $\sigma = \sigma_2$ and set $f_i = p_i^{-1} f$ for $i = 1, \dots, k$. Set $f_i = 0$ if $f \in \Bbbk$.

Suppose σ^n is an inner automorphism of $\mathcal{R}$ for some integer n. Then there exists a unit $\eta \in \mathcal{R}$ such that $\eta^{-1} r \eta = \sigma^n(r)$ for all $r \in \mathcal{R}$. Since η is a unit, then (up to a scalar) $\eta = p_1^{m_1} \cdots p_k^{m_k}$ for some $m_i \in \mathbb{Z}$. Recall,

$$p_i^{-1} a p_i = p_i^{-1}(p_i a + f) = a + f_i.$$

By Proposition 5.5, $\sigma(a) = a - g$ and

$$a - ng = \sigma^n(a) = \eta^{-1} a \eta = a + \sum_{i=1}^{k} m_i f_i.$$

Thus, $g = -\frac{1}{n} \sum_{i=1}^{k} m_i f_i$. The converse is similar. $\square$

Corollary 6.6. *If $f, g \in \Bbbk[c]$ are such that g is not a local reduction of f, then the height one primes of P are of the form (u) and $(c - \alpha)$ for $\alpha \in \Bbbk$ with $f(\alpha) = 0$. Moreover, $P/(c - \alpha) \cong \Bbbk[a, u]$ and $P/(u) \cong R_f$.*

In case g is not a local reduction of f, the finite-dimensional representation theory of $P = P(f, g)$ is easy to describe in light of Corollary 6.6. In particular, for every $\alpha, \beta \in \Bbbk$ with $f(\alpha) = 0$, there exists a 1-dimensional (irreducible) P-module $M = \{m\}$ such that $u.m = 0$, $c.m = \alpha m$ and $a.m = \beta m$. On the other hand, for every irreducible polynomial $\Omega \in \Bbbk[a, u]$ of degree n, there exists an irreducible n-dimensional representation.

Proposition 6.7. *Choose $f, g \in \Bbbk[c]$ and set $P = P(f, g)$. The algebra P is a noetherian domain with* $\operatorname{GKdim} P = 3$ *and*

$$\operatorname{gldim} P = \begin{cases} 2 & \text{if } f \in \Bbbk \\ 3 & \text{otherwise.} \end{cases}$$

Proof. It follows easily that P is a noetherian domain from the skew polynomial ring construction. The ring $\Bbbk[c, u]$ has GK and global dimension two. Using Presentation 2, the result on GK dimension for P now follows by [17, Corollary 8.2.11].

By [17, Theorem 7.10.3], $\operatorname{gldim} P = 3$ if P has a D-stable prime of height two and otherwise $\operatorname{gldim} P = 2$.

Suppose $f \in \mathbb{k}$. If $g \neq 0$, then by Proposition 6.5, (u) is the unique maximal ideal of P and it has height one. If $g = 0$, then $P \cong A_1(\mathbb{k})[u]$ and by the simplicity of $A_1(\mathbb{k})$, the maximal ideals are of the form $(u - \beta)$, $\beta \in \mathbb{k}$, and they each have height one.

Now suppose $f \notin \mathbb{k}$. As in Proposition 6.5, let $\alpha \in \mathbb{k}$ such that $f(\alpha) = 0$ and set $f_0 = (c-\alpha)^{-1} f$. Since $f \in I_3$, the calculation below shows $D(ur + (c-\alpha)s) \in I_3$ for $r, s \in P$.

$$\begin{aligned} D(ur + (c-\alpha)s) &= uD(r) + D(u)r + (c-\alpha)D(s) + D(c-\alpha)s \\ &= u\left(D(r) + gr\right) + (c-\alpha)\left(D(s) + f_0 s\right). \end{aligned}$$

Therefore, I_3 is a D-stable prime of height two in P and we may conclude P has global dimension three. $\square$

The base ring of the Jordanian matrix ring $M_J(2)$ is $P(c^2, c)$ and, in this case, g is a local reduction of f. More generally, if $f = cg$, then g is a local reduction of f since c is an irreducible factor of f and $g = c^{-1} f$. In this case the prime ideal spectrum must be more complicated since Q (as defined in Proposition 6.5) is not simple. In the ISPE $P(cg, g)$, the element $u - \beta c$ is normal for any $\beta \in \mathbb{k}$. We wish to show that the remaining ideals are those of the form $(u - \beta c)$.

Proposition 6.8. *Let $P = P(f,g)$ for some $f, g \in \mathbb{k}[c]$ such that f is monic. Set $\mathcal{P} = P(f,g)\mathcal{C}^{-1}$, where $\mathcal{C}$ is the Ore set generated by powers of the monic irreducible factors $p_1, \dots, p_k$ of f. If g is a local reduction of f, then set $g = -\frac{1}{n}\sum_{i=1}^{k} m_i f_i$ with $m_1, \dots, m_k \in \mathbb{Z}$, $n \in \mathbb{N}$, and $f_i = {p_i}^{-1} f$.*

(1) *If $\mathcal{Z}(\mathcal{P}) \neq \mathbb{k}$, then g is not a local reduction of f.*
(2) *If g is a local reduction of f, then $\mathcal{Z}(\mathcal{P}) = \mathbb{k}[v]$ with $v = -\left(p_1^{-m_1} \cdots p_k^{-m_k}\right) u^n$.*
(3) *The center of $P(f,g)$ is $\mathbb{k}$.*

Proof. We only need to prove (2) since (1) follows from Proposition 6.5 and (3) follows from (1) and (2).

A direct calculation shows $ap_i = p_i\left(a + f_i\right)$ for all $i \leq k$. Moreover,

$$a\left(p_1^{-m_1} \cdots p_k^{-m_k}\right) = \left(p_1^{-m_1} \cdots p_k^{-m_k}\right)\left(a + \sum_{i=1}^{k} -m_i f_i\right)$$

and

$$au^n = u^n\left(a + ng\right) = u^n\left(a + \sum_{i=1}^{k} -m_i f_i\right).$$

Putting these two equations together gives

$$a\left(p_1^{-m_1} \cdots p_k^{-m_k}\right)u^n = \left(p_1^{-m_1} \cdots p_k^{-m_k}\right)u^n a,$$

hence $\left(p_1^{-m_1} \cdots p_k^{-m_k}\right)u^n$ is central. This proves v is central. To finish the proof of (2), we must show any central element of $\mathcal{P}$ is a linear combination of powers of v.

Given $\omega \in \mathcal{Z}(\mathcal{P})$ we may write $\omega = \sum r_i a^i$ with $r_i \in \mathbb{k}[c,u]\mathcal{C}^{-1}$. Since ω is central, then

$$0 = [\omega, c] = \sum r_i [a^i, c] = \sum r_i \left((a+g)^i - a^i\right).$$

Hence, the a-degree of ω is zero. We assume $\omega \notin \Bbbk$ and write

$$\omega = \sum_{i=0}^{m} \omega_i u^i.$$

with $\omega_0, \ldots, \omega_m \in \Bbbk[c]\mathcal{C}^{-1}$ such that $\omega_m \neq 0$, $m \geq 1$. Since $a\omega = \omega a$, then $D\omega = 0$. Thus,

$$0 = D\omega = \sum_{i=0}^{m} \left((D\omega_i)\, u^i + \omega_i \left(Du^i \right) \right) = \sum_{i=0}^{m} \left(f\left(\partial_c \omega_i\right) u^i + ig\omega_i u^i \right) = \sum_{i=0}^{m} \left(f\left(\partial_c \omega_i\right) + ig\omega_i \right) u^i.$$

According to the above calculation, $f\left(\partial_c \omega_i\right) + ig\omega_i = 0$ for $i = 0, \ldots, m$. We may use separation of variables to solve this differential equation. This gives $\omega_i = \lambda_i \left(p_1^{-m_1} \cdots p_k^{-m_k} \right)^{i/n}$ for some $\lambda_i \in \Bbbk$. If $\lambda_i \neq 0$, then the power i/n must be a nonnegative integer since $\omega_i \in \Bbbk[c]\mathcal{C}^{-1}$. This finishes the proof of (2) since

$$\omega = \sum_{i=0}^{m} \lambda_i \left(p_1^{-m_1} \cdots p_k^{-m_k} \right)^{i/n} u^i \ = \sum_{i=0}^{m} (-1)^{i/n} \lambda_i v^{i/n} \in \Bbbk[v].$$

□

Remark 6.9. *In the special case $f = cg$, we have $v = c^{-1}u$.*

Fix $f \in \Bbbk[c]$. If $f \notin \Bbbk$, then up to isomorphism of $P(f, g)$, we may choose f such that $f(0) = 0$. *For the remainder of this section, we set $g = c^{-1}f$ and let $P = P(cg, g)$ when $f \notin \Bbbk$.* This hypothesis implies that g is a local reduction of f. Hence, $\mathcal{Z}(\mathcal{P}) = \Bbbk[c^{-1}u]$. Set $v = c^{-1}u$. If $f \in \Bbbk$, we set $f = 1$ and $g = 0$.

Proposition 6.10. *If $f \notin \Bbbk$, then the height one primes of P are of the form $(c - \alpha)$ and $(u - \beta c)$ for $\alpha, \beta \in \Bbbk$ with $f(\alpha) = 0$ and β arbitrary. Moreover, $P/(c - \alpha)P \cong \Bbbk[a, u]$ and $P/(u - \beta c)P \cong R_f$.*

Proof. Since $\mathcal{R}$ is simple, then by [15, Corollary 2.3] the prime ideals of P are in 1-1 correspondence with $\mathcal{Z}(\mathcal{P}) = \Bbbk[v]$. Hence, the remaining prime ideals are generated in $\mathcal{P}$ by $v - \beta$ and $(v - \beta) \cap P = (u - \beta c)$. □

There is another way to view the above proposition. Note that $P[g^{-1}]$ is isomorphic to the universal enveloping algebra of the 3-dimensional Lie algebra L on generators x, y, z subject to the relations $[x, y] = y$, $[x, z] = z$, and $[y, z] = 0$. Hence, the prime ideals of P disjoint from irreducible factors of g are in $1-1$ correspondence with the prime ideals of $U(L)$.

Proposition 6.11. *Suppose $f \notin \Bbbk$. For any $\alpha, \lambda, \mu \in \Bbbk^\times$, $\eta \in \Bbbk$, and $h \in \Bbbk[u, c]$ there is an automorphism π of P such that $\pi(g) = \lambda g$ determined by the assignments*

$$a \mapsto \lambda a + h, \quad c \mapsto \varepsilon c, \quad \text{and } u \mapsto \mu u + \eta c.$$

Moreover, any automorphism of P has the above form.

Proof. Let $\pi \in \mathrm{Aut}(P)$. By Proposition 6.10, π fixes the ideals $(c - \alpha)$ and $(u - \beta c)$. Hence $\pi(c) = \varepsilon c + \kappa$ for some $\varepsilon \in \Bbbk^\times$ and $\kappa \in \Bbbk$ with $f(-\varepsilon^{-1}\kappa) = 0$. Similarly, $\pi(u) = \mu u + \eta c$ for some $\mu \in \Bbbk^\times$ and $\eta \in \Bbbk$. A degree counting argument now shows that $\pi(a) = \lambda a + h$ for some $\lambda \in \Bbbk^\times$ and $h \in \Bbbk[c, u]$. To complete the proof we must show $\kappa = 0$, $\pi(g) = \lambda g$, and $\pi(f) = \varepsilon \lambda f$.

Applying π to $au = ua + ug$ gives $\lambda\mu gu + \lambda\eta f = \mu\pi(g)u + \eta\pi(g)c$. Equating coefficients of u gives $\lambda\mu gu = \mu\pi(g)u$ and $\lambda\eta f = \eta\pi(g)c$. Thus $\pi(g) = \lambda g$. Applying π to $ac = ca + f$ and reducing gives $\pi(f) = \varepsilon\lambda f$. Combining this with $f = cg$ and $\pi(g) = \lambda g$ gives $\kappa = 0$. $\square$

The following proposition is now a consequence of Proposition 4.5.

Proposition 6.12. *The ISPE P is birationally equivalent to $A_1(\Bbbk[t])$.*

7. Generalized Jordanian Matrix Algebras

Definition 7.1. *We say $M = P(f,g)[b;\sigma_3,\delta_3]$ is a Generalized Jordanian Matrix Algebra (GJMA) if it is birationally equivalent to $M_J(2)$ and the involution τ extends to M with $\tau(b) = b$.*

Remark 7.2. *By* [9, Proposition 1.8]*, $M_J(2)$ is birationally equivalent to $A_1(\Bbbk[s,t])$. Thus, $P(f,g)[b;\sigma_3,\delta_3]$ is a GJMA if and only if it is birationally equivalent to $A_1(\Bbbk[s,t])$ and τ extends appropriately.*

Example 7.3. *Since the ISPE $P(1,0) \cong A_1(\Bbbk[t])$, then it follows that $A_1(\Bbbk[s,t]) \cong P(1,0)[b]$ is itself a GJMA.*

In this section, we construct a family of GJMAs and study their properties. We fix some notation throughout this section. Let $f, g \in \Bbbk[c]$, $f \notin \Bbbk$, such that g is a local reduction of f and set $P = P(f,g)$. Let v be the central element defined in Proposition 6.8. Let $\mathcal{C}$ be the Ore set generated by powers of irreducible factors of f and $\mathcal{U}$ the Ore set generated by $\mathcal{C}$ along with powers of $u - \beta c$ for $\beta \in \Bbbk$. Let $\mathcal{P} = P\mathcal{C}^{-1}$ and $\mathscr{P} = P\mathcal{U}^{-1}$. To simplify notation, we write $\sigma = \sigma_3$ and $\delta = \delta_3$. In order to construct GJMAs $M = P[b;\sigma,\delta]$ we consider restrictions on σ and δ. Let $\mathcal{M} = M\mathcal{C}^{-1}$ and $\mathscr{M} = M\mathcal{U}^{-1}$. We rely on comparisons with $M_J(2)$ and key characteristics of P outlined in Section 6 to identify additional restrictions.

Proposition 7.4. *If σ is an inner automorphism of $\mathscr{P}$ and δ is an inner σ-derivation, then $M = P[b;\sigma,\delta]$ is birationally equivalent to $M_J(2)$.*

Proof. By the hypothesis, there exist $\varphi \in \mathcal{C}$ and $\theta \in \mathscr{P}$ such that $\sigma(r) = \varphi^{-1}r\varphi$ and $\delta(r) = \theta r - \sigma(r)\theta$ for all $r \in \mathscr{P}$. In particular, the element $z = \varphi(b - \theta)$ is central in $\mathscr{M}$. Moreover, $v = c^{-1}u$ is central in $\mathcal{P}$ by Proposition 6.8. Thus, it is clear $\{a, c, v, z\}$ is a $\Bbbk$-algebra generating set of $\mathscr{M}$. Moreover, there are inverse homomorphisms $\Phi : \mathscr{M} \to \mathrm{Frac}(A_1(\Bbbk[s,t]))$ and $\Psi : \mathrm{Frac}(A_1(\Bbbk[s,t])) \to \mathscr{M}$ given by

$$\Phi(a) = xf(y), \Phi(c) = y, \Phi(v) = yt, \Phi(z) = s,$$

$$\Psi(x) = af^{-1}(c), \Psi(y) = c, \Phi(t) = v, \Phi(s) = z.$$

Thus, M is birationally equivalent to $M_J(2)$. $\square$

Example 7.5. *Let $P = P(c, 1)$ and consider the algebra $G = P[b; \sigma, \delta]$ given by*

$$\sigma(a) = c^{-1}ac = a + 1, \;\; \sigma(u) = u, \;\; \sigma(c) = c,$$

$$\delta(a) = -(u/2 + a)a - (a + 1)(-(u/2 + a)) = (-ua/2 - a^2) + (au/2 + a^2 + u/2 + a) = u + a,$$

$$\delta(u) = \tau(\delta(a)) - \delta(a) = -u + u + a - (u + a) = -u,$$

$$\delta(c) = \theta c - \sigma(c)\theta = -(u/2 + a)c - c(-(u/2 + a)) = -ac + ca = -c.$$

Thus, the relations for G may be given as those for $P(c, 1)$ along with

$$bc = cb - c, \;\; bu = ub - u, \;\; ba = (a + 1)b + (u + a).$$

The element $z = 2(cb + ca) + uc$ is central in G. It is not difficult to show that G satisfies Proposition 7.4 and that $\tau(b) = b$ extends to an involution of G. Hence, G is a GJMA.

We now restrict to the case $f = cg$ so that $v = c^{-1}u$.

Proposition 7.6. *Choose $f \in \Bbbk[c]$, $f = cg$, and let $P = P(f, g)$. Choose p an irreducible factor of f and $\theta \in P$. Define $G(f, p, \theta) := P[b; \sigma, \delta]$ by $\sigma(x) = p^{-1}xp$ and $\delta(x) = \theta x - \sigma(x)\theta$ for $x = a, u, c$. Then $G(f, p, \theta)$ is a GJMA if and only if $\delta(u) = \tau(\delta(a)) - \delta(a)$ and $\delta(c) = \tau(\delta(c))$.*

Proof. By construction, σ is an inner automorphism of P. Note that $p^{-1}cp = c$, $p^{-1}up = u$, and $p^{-1}ap = p^{-1}p'f \in P$ since p is a factor of f. Moreover, δ is the inner σ-derivation of P determined by θ.

By Proposition 7.4, $G(f, p, \theta)$ is a GJMA if τ extends to an involution with $\tau(b) = b$. We must show τ is a homomorphism of G. It will then follow that τ is an involution of G.

Applying τ to the identity $0 = \sigma(a)b - ba + \delta(a)$ gives

$$\begin{aligned} 0 &= \left(p^{-1}(u + a)p\right) b - b(u + a) + \tau(\delta(a)) \\ &= (\sigma(a)b - ba) + (\sigma(u)b - bu) + \tau(\delta(a)) \\ &= \sigma(u)b - bu + (\tau(\delta(a)) - \delta(a)) . \end{aligned}$$

This identity is fixed by τ and hence holds if and only if $\delta(u)$ satisfies the hypothesis. Repeating with $bc = cb + \delta(c)$ gives $\tau(\delta(c)) = \delta(c)$. □

Proposition 7.7. *Suppose $G = G(f, p, \theta)$ is a GJMA. We localize G by localizing the base ring P and extending σ and δ appropriately. We write $\mathcal{G} = \mathcal{P}[b; \sigma, \delta]$. Then $v = c^{-1}u$ is central in $\mathcal{G}$ if and only if $\delta(c) = cu^{-1}\delta(u)$.*

Proof. By Proposition 6.8, $v = c^{-1}u$ is central in $\mathcal{P}$. We have

$$bv = \left(c^{-1}b - c^{-1}\delta(c)c^{-1}\right) u = c^{-1}(ub + \delta(u)) - c^{-1}\delta(c)c^{-1}u = vb + c^{-1}(\delta(u) - \delta(c)v) .$$

Hence, v is central in $\mathcal{G}$ if and only if $\delta(u) = \delta(c)v$ or, equivalently, $c\delta(u) = u\delta(c)$. Because c and u commute, then $\delta(c) = cu^{-1}\delta(u)$. This formula must agree with $\theta c - c\theta$. Recall $\delta(u) = \tau(\delta(a)) - \delta(a)$ and $\delta(u) = u\gamma$, hence $\gamma = u^{-1}(\tau(\delta(a)) - \delta(a))$ so $\delta(c) = cu^{-1}\delta(u) = cu^{-1}(\tau(\delta(a)) - \delta(a))$. $\square$

Suppose $\deg g \geq 1$. Set $h = g'c$ and $\theta = g^{-1}\left(a^2 + (u-g)a\right)$. Then we have

$$\begin{aligned}
\theta(a) - (a+h)\theta &= g^{-1}\left[\left(a^2 + (u-g)a\right)a - a\left(a^2 + (u-g)a\right)\right] \\
&= g^{-1}\left[(ua - ga)a - (ua + ug - ga - g'f)a\right] \\
&= g^{-1}\left[-(ug - g'f)a\right] = (h-u)a, \\
\theta(c) - (c)\theta &= g^{-1}\left[\left(a^2 + (u-g)a\right)c - c\left(a^2 + (u-g)a\right)\right] \\
&= g^{-1}\left[\left(a + (u-g)\right)c(a+g) - c\left(a^2 + (u-g)a\right)\right] \\
&= cg^{-1}\left[\left((a+g) + (u-g)\right)(a+g) - \left(a^2 + (u-g)a\right)\right] \\
&= cg^{-1}\left[\left(a^2 + ag + ua + ug\right) - \left(a^2 + (u-g)a\right)\right] \\
&= cg^{-1}\left[ga + g'f + ug + ga\right] = c\left(h + u + 2a\right).
\end{aligned}$$

The computation for u is similar.

Proposition 7.8. *The ring $G\left(f, g, g^{-1}\left(a^2 + (u-g)a\right)\right)$ is a GJMA which we denote by $\mathcal{G}_f$. Moreover, $\mathcal{G}_{c^2} = M_J(2)$.*

Proof. We have $g^{-1}ag = a + h$. Set $\gamma = (h + u + 2a)$. By Proposition 7.6, it suffices to show the following,

$$\begin{aligned}
\tau(\delta(a)) - \delta(a) = \tau((h-u)a)) - (h-u)a &= (h+u)(u+a) - (h-u)a \\
&= (hu + ha + u^2 + ua) - (ha - ua) = hu + u^2 + 2ua = u\gamma = \delta(u).
\end{aligned}$$

and $cu^{-1}\delta(u) = c\gamma = \delta(c)$. $\square$

The relations for $\mathcal{G}_f$ can be shown to be

$$\begin{aligned}
&ac = ca + f, au = ua + ug, cu = uc, \\
&bc = cb + c\gamma, bu = ub + u\gamma, ba = (a+h)b + (h-u)a.
\end{aligned}$$

Explicitly, σ and δ are given by

$$\begin{aligned}
&\sigma(c) = c, \sigma(u) = u, \sigma(a) = a + h, \\
&\delta(c) = c\gamma, \delta(u) = u\gamma, \delta(a) = (h-u)a.
\end{aligned}$$

By construction, the element $p(b-\theta)$ lies in the center of any $G(f, p, \theta)$. The analog of the determinant in the GJMA G_f is the central element $z = gb + (g-u)a - a^2$.

We may also write the presentation in terms of the standard generators a, b, c, d. In this case, $\gamma = h+a+d$ and we have relations

$$ac = ca + f, dc = cd + f, da = (a - g)d + g(a - h),$$
$$bc = cb + c\gamma, ba = (a + h)b + (h - u)a, bd = (d + h)b + (h - u)d.$$

Proposition 7.9. *The algebra $\mathcal{G}_f$ is a noetherian domain with* $\operatorname{gldim} \mathcal{G}_f = \operatorname{GKdim} \mathcal{G}_f = 4$. *Moreover, the center of $\mathcal{G}_f$ is $\Bbbk[z]$.*

Proof. That $\mathcal{G}_f$ is a noetherian domain follows from the skew polynomial construction.

The statement on the center follows in similar fashion to [9, Proposition 1.8]. In $\mathcal{G}_f\mathcal{C}^{-1}$ we have $b = g^{-1}\left(z - (g - u)a + a^2\right)$. Let $R = \Bbbk[z, v]\mathcal{C}^{-1}$, then $\mathcal{G}_f\mathcal{C}^{-1}$ is isomorphic to $S = R[a, D]$ where D is extended from Presentation 2 of Remark 6.2. In particular, $D(z) = D(v) = 0$. In S, $[c^{-1}, a] = gc^{-1}$ and an induction argument shows that $[c^{-1}, a^n] = nga^{n-1} +$ (lower a-degree terms). If $s = \sum_{i=0}^{n} r_i a^i \in S$ with $r_i \in R$ for all i, then $[c^{-1}, s] = \sum_{i=0}^{n} r_i[c^{-1}, a^i] = r_n nga^{n-1} +$ (lower a-degree terms). Hence, if $s \in \mathcal{Z}(S)$, then $s \in R$. Since $D(s) = [a, s] = 0$, then $s \in \Bbbk[z, v]$. Thus, $\mathcal{Z}(S) = \Bbbk[z, v]$ and $\mathcal{Z}(\mathcal{G}_f) = \mathcal{Z}(S) \cap \mathcal{Z}(\mathcal{G}_f) = \Bbbk[z]$.

By [17, Theorem 7.5.3 (i)] and Proposition 6.3, $\operatorname{gldim} \mathcal{G}_f \leq 1 + \operatorname{gldim} P = 4$. On the other hand, $P[z]$ is faithfully flat as an $\mathcal{G}_f$-module. Thus, by [17, Theorem 7.2.6] $\operatorname{gldim} \mathcal{G}_f \geq \operatorname{gldim} P[z] = 4$.

Since $V = \{a, c, u\}$ is a finite-dimensional generating subspace of P and $\sigma(V) \subseteq V$, then by [13, Lemma 2.2] and Proposition 6.3, $\operatorname{GKdim} \mathcal{G}_f = 1 + \operatorname{GKdim} P = 4$. □

Proposition 7.10. *The height one prime ideals of $\mathcal{G}_f$ are (c), (u), and $(z - \xi)$ for $\xi \in \Bbbk^\times$.*

Proof. The height one prime ideals of P are principally generated by $c - \alpha$ and $u - \beta c$ where $\alpha, \beta \in \Bbbk$ with $f(\alpha) = 0$ and β arbitrary (Proposition 6.10). Then $\mathscr{P}$ is simple by Lemma 6.10.

In $\mathcal{G}_f$, $c - \alpha$ (resp. $u - \beta c$) is a normal element if and only $\alpha = 0$ (resp. $\beta = 0$). Hence, the height one prime ideals of $\mathcal{G}_f$ that have nonzero intersection with P are (u) and (c). By Proposition 7.9, the center of $\mathcal{G}_f$ is $\Bbbk[z]$. Hence, the ideals of $\mathscr{P}$ which lie over 0 in P are of the form $(z - \xi)$ for $\xi \in \Bbbk^\times$ [15, Corollary 2.3]. □

Remark 7.11. *We think of $S = \mathcal{G}_f/(z - 1)$ as an analog of* $\mathrm{SL}_J(2)$*, the Jordanian deformation of* $\mathrm{SL}(2)$*. Because $(z - 1)$ is irreducible in $\mathcal{G}_f$, $(z - 1)$ is a completely prime ideal. That is, S is an integral domain.*

References

[1] J. Alev and M. Chamarie. Dérivations et automorphismes de quelques algèbres quantiques. *Comm. Algebra*, 20(6):1787–1802, 1992.

[2] J. Alev and F. Dumas. Rigidité des plongements des quotients primitifs minimaux de $U_q(\mathrm{sl}(2))$ dans l'algèbre quantique de Weyl-Hayashi. *Nagoya Math. J.*, 143:119–146, 1996.

[3] J. Alev and F. Dumas. Invariants du corps de Weyl sous l'action de groupes finis. *Comm. Algebra*, 25(5):1655–1672, 1997.

[4] M. Awami, M. Van den Bergh, and F. Van Oystaeyen. Note on derivations of graded rings and classification of differential polynomial rings. *Bull. Soc. Math. Belg. Sér. A*, 40(2):175–183, 1988. Deuxième Contact Franco-Belge en Algèbre (Faulx-les-Tombes, 1987).

[5] Ken A. Brown and Ken R. Goodearl. *Lectures on algebraic quantum groups*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2002.

[6] E. E. Demidov, Yu. I. Manin, E. E. Mukhin, and D. V. Zhdanovich. Nonstandard quantum deformations of $\mathrm{GL}(n)$ and constant solutions of the Yang-Baxter equation. *Progr. Theoret. Phys. Suppl.*, (102):203–218 (1991), 1990. Common trends in mathematics and quantum field theories (Kyoto, 1990).

[7] Warren Dicks. Automorphisms of the polynomial ring in two variables. *Publ. Sec. Mat. Univ. Autònoma Barcelona*, 27(1):155–162, 1983.

[8] Jacques Dixmier. Sur les algèbres de Weyl. *Bull. Soc. Math. France*, 96:209–242, 1968.

[9] François Dumas and Laurent Rigal. Prime spectrum and automorphisms for 2×2 Jordanian matrices. *Comm. Algebra*, 30(6):2805–2828, 2002.

[10] Jason Gaddis. Isomorphisms of some quantum spaces. In *Ring theory and its applications*, volume 609 of *Contemp. Math.*, pages 107–116. Amer. Math. Soc., Providence, RI, 2014. Appendix at arXiv:11210.8413.

[11] Jason Gaddis. Two-generated algebras and standard-form congruence. *Comm. Algebra*, 43(4):1668–1686, 2015.

[12] K. R. Goodearl and R. B. Warfield, Jr. *An introduction to noncommutative Noetherian rings*, volume 61 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, second edition, 2004.

[13] Chan Huh and Chol On Kim. Gelfand-Kirillov dimension of skew polynomial rings of automorphism type. *Comm. Algebra*, 24(7):2317–2323, 1996.

[14] T. Y. Lam. *The algebraic theory of quadratic forms*. W. A. Benjamin, Inc., Reading, Mass., 1973. Mathematics Lecture Note Series.

[15] André Leroy and Jerzy Matczuk. Prime ideals of Ore extensions. *Comm. Algebra*, 19(7):1893–1907, 1991.

[16] Yuri I. Manin. *Topics in noncommutative geometry*. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1991.

[17] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings*, volume 30 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, revised edition, 2001. With the cooperation of L. W. Small.

[18] S. P. Smith. Quantum groups: an introduction and survey for ring theorists. In *Noncommutative rings (Berkeley, CA, 1989)*, volume 24 of *Math. Sci. Res. Inst. Publ.*, pages 131–178. Springer, New York, 1992.

[19] S. Paul Smith and Michel Van den Bergh. Noncommutative quadric surfaces. *J. Noncommut. Geom.*, 7(3):817–856, 2013.

[20] James J. Zhang and Jun Zhang. Double Ore extensions. *J. Pure Appl. Algebra*, 212(12):2668–2690, 2008.

[21] James J. Zhang and Jun Zhang. Double extension regular algebras of type (14641). *J. Algebra*, 322(2):373–409, 2009.

(Gaddis) Wake Forest University, Department of Mathematics at Statistics, P. O. Box 7388, Winston-Salem, NC 27109

E-mail address: gaddisjd@wfu.edu

(Price) University of Wisconsin - Oshkosh, Department of Mathematics, 800 Algoma Blvd., Oshkosh, WI 54901

E-mail address: pricek@uwosh.edu